# Math 218

## Solving Session (Sat, Apr. 21, 2007)

**page 227 ex. 12**

$V = \left\{ \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} \text{ where } a \text{ and } b \text{ are scalars} \right\}$

Standard Addition and scalar multiplication

Take $u, v \in V$ so $u = \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix}$ and $v = \begin{bmatrix} x' & x'+y' \\ x'+y' & y' \end{bmatrix}$

**Axiom 1:**

$$u+v = \begin{bmatrix} x+x' & (x+y)+(x'+y') \\ (x+y)+(x'+y') & y+y' \end{bmatrix} = \begin{bmatrix} x+x' & (x+x')+(y+y') \\ (x+x')+(y+y') & y+y' \end{bmatrix}$$

Therefore $u+v$ has the form of elements of $V$ so $u+v \in V$

**Axiom 2:**

$$u+v = \begin{bmatrix} x+x' & (x+y)+(x'+y') \\ (x+y)+(x'+y') & y+y' \end{bmatrix} \qquad v+u = \begin{bmatrix} x'+x & (x'+y')+(x+y) \\ (x'+y')+(x+y) & y'+y \end{bmatrix}$$

so $u+v = v+u$ from the commutativity of addition in $\mathbb{R}$.

**Axiom 3:**
similar to 2.

**Axiom 4:** Is there $0_V \in V$ such that $u + 0_V = u$?
If so, then $0_V = \begin{bmatrix} c & c+d \\ c+d & d \end{bmatrix}$ for some $c$ and $d$

$$u + 0_V = u \Rightarrow \begin{bmatrix} x+c & (x+y)+(c+d) \\ (x+y)+(c+d) & y+d \end{bmatrix} = \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix}$$

$$\Rightarrow \begin{array}{c} x+c = x \\ y+d = y \\ (x+y)+(c+d) = x+y \end{array} \Rightarrow \begin{cases} c = 0 \\ d = 0 \end{cases}$$

Therefore, $0_V = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Axiom 5: check that $-u = \begin{bmatrix} -x & -(x+y) \\ -(x+y) & -y \end{bmatrix}$

when $u = \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix}$

Axiom 6:
$k \in \mathbb{R}$ $\quad ku = \begin{bmatrix} kx & k(x+y) \\ k(x+y) & ky \end{bmatrix} = \begin{bmatrix} kx & kx+ky \\ kx+ky & ky \end{bmatrix} \in V$

Axiom 7:

Continue as before

Note: If $V = \left\{ \begin{bmatrix} a & ab \\ ab & b \end{bmatrix} \text{ where } a \text{ \& } b \text{ are scalars} \right\}$

then for example axiom 6 doesn't work since

$ku = k\begin{bmatrix} x & xy \\ xy & y \end{bmatrix} = \begin{bmatrix} kx & kxy \\ kxy & ky \end{bmatrix} \notin V$

If it were in V then instead of $kxy$ we should have $k^2xy$.

---

page 228 ex. 20

It is not a vector space since if we take

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ A is a 2x2 invertible matrix

$B = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$ B is also a 2x2 invertible matrix

But $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ not invertible

so Axiom 1 doesn't hold.

(Note: Notice that also Axiom 4 doesn't hold since $0_V = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not in the vector space)

page 238 ex. 5

(d) W= nxn matrices A such that AB=BA for a fixed nxn matrix B

$W = \{ A \mid AB = BA \}$

To see if it is a subspace check

1. if $u, v \in W$ then $u+v \in W$?

u and v are in W then $u = A_1$ where $A_1 B = BA_1$

$v = A_2$ where $A_2 B = BA_2$

$u+v = A_1 + A_2$.

Is $u+v$ in W? (i.e is $u+v$ an nxn matrix such that $(u+v)B = B(u+v)$)

True since $(A_1 + A_2)B = A_1 B + A_2 B = BA_1 + BA_2$
$= B(A_1 + A_2)$

2. $k \in \mathbb{R}$, $u \in W$ is $ku \in W$?

$ku = kA_1$ is an nxn matrix

$(kA_1)B = kA_1 B = B(kA_1)$

Therefore, $ku \in W$.

So, W is a subspace.

---

page 239 ex. 6

(a) $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix}$

Solve the system $Ax = 0$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & -4 & -5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & -2 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $x_3 = t$

$x_2 = -\frac{3}{2}t$

$x_1 = t - \frac{3}{2}t = -\frac{1}{2}t$

$$\text{Solution Space} = \{ x \in \mathbb{R}^3 \mid Ax = 0 \}$$

$$= \left\{ x \in \mathbb{R}^3 \ \left( x = (x_1, x_2, x_3) \text{ or } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) \ \middle| \ \begin{matrix} x_1 = -\frac{1}{2} t \\ x_2 = -\frac{3}{2} t \\ x_3 = t \end{matrix} \right\}$$

So the solution space is a line with parametric equation

$$\begin{cases} x = -\frac{1}{2} t \\ y = -\frac{3}{2} t \\ z = t \end{cases} \qquad t \in \mathbb{R}$$

Notice that I renamed $x_1, x_2, x_3$ by $x, y, z$. Of course, you can also leave them.

---

page 239 ex. 9

(a) $p_1 = 2 + x + 4x^2$

$p_2 = 1 - x + 3x^2$

$p_3 = 3 + 2x + 5x^2$

Can $-9 - 7x - 15x^2$ be equal to $k_1 p_1 + k_2 p_2 + k_3 p_3$ for some $k_1, k_2, k_3$?

$$\begin{aligned} -9 - 7x - 15x^2 &= k_1 p_1 + k_2 p_2 + k_3 p_3 \\ &= k_1 (2 + x + 4x^2) + k_2 (1 - x + 3x^2) + k_3 (3 + 2x + 5x^2) \\ &= (2k_1 + k_2 + 3k_3) + (k_1 - k_2 + 2k_3)x + (4k_1 + 3k_2 + 5k_3)x^2 \end{aligned}$$

$$\begin{cases} 2k_1 + k_2 + 3k_3 = -9 \\ k_1 - k_2 + 2k_3 = -7 \\ 4k_1 + 3k_2 + 5k_3 = -15 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & -7 \\ 2 & 1 & 3 & -9 \\ 4 & 3 & 5 & -15 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & -7 \\ 0 & 3 & -1 & 5 \\ 0 & 7 & -3 & 13 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & -7 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & -2/3 & 4/3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & -7 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

so $k_3 = -2$ $\quad k_2 = \frac{5}{3} - \frac{2}{3} = \frac{3}{3} = 1$ $\quad k_1 = -7 + 4 + 1 = -2$

page 240 ex. 23

(a) False if $b \neq 0$

$x_1$ is a solution for $Ax = b$ so $Ax_1 = b$

$x_2$ is a solution for $Ax = b$ so $Ax_2 = b$

$A(x_1 + x_2) = 2b$

so $x_1 + x_2$ is a solution to the system $Ax = 2b$.

(b) True For $k = 1$ we get $u + v \in W$ for all $u, v \in W$

for $v = 0$ we get $ku \in W$ for all $u \in W$.

(c) True

(d) True <u>Want:</u> $W_1 \cap W_2$ is a subspace of $V$

<u>Given:</u> $W_1$ is a subspace of $V$

$W_2$ is a subspace of $V$

<u>want:</u> if $u, v \in W_1 \cap W_2$, $k$ is a scalar then $u + v \in W_1 \cap W_2$, $ku \in W_1 \cap W_2$

$u \in W_1 \cap W_2 \implies u \in W_1$ and $u \in W_2$

$v \in W_1 \cap W_2 \implies v \in W_1$ and $v \in W_2$

But $W_1$ is a subspace so $u + v \in W_1$, $ku \in W_1$

and $W_2$ is also a subspace so $u + v \in W_2$, $ku \in W_2$

Therefore $u + v \in W_1 \cap W_2$

$ku \in W_1 \cap W_2$

(e) False counterexample

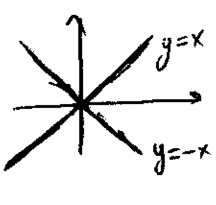
$\mathrm{Span}\{(1,1)\} = \mathrm{Span}\{(1,1), (2,2)\}$

<u>Note:</u> if $W_1$ and $W_2$ are subspaces, $W_1 \cup W_2$ need not be a subspace.

Take for example, $W_1$ is the line $y = x$

$W_2$ is the line $y = -x$

Both are subspaces but $W_1 \cup W_2$ is not.



page 249 ex. 11

If $S = \{v_1, v_2, --- v_r\}$ is linearly independent
$\Leftrightarrow$ no element can be written as a linear combination of the others.

So take any subset $S'$ of $S$, no element can be written as a linear combination of the others
$\Leftrightarrow$ $S'$ is linearly independent.

2nd method:

By contradiction,

Say there is a subset $S' = \{v_1, -- v_i\}$ of $S$ that is linearly dependent. (without loss of generality)

So, there is a non trivial solution $(c_1^*, -- c_i^*)$ for the equation $c_1 v_1 + --- c_i v_i = 0$

But then $(c_1^*, -- c_i^*, 0, -- 0)$ is a non trivial solution for the equation

$$c_1 v_1 + -- c_i v_i + -- c_r v_r = 0$$

So $S$ is linearly dependent which is not true.

Therefore $S'$ is linearly independent.